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Nonlinear Second Order Equations with Applications to Partial Differential Equations

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1. INTRODUCTION

In this paper we study the Cauchy problem for the abstract second order (in time) semilinear differential equation

$$u''(t) + Au'(t) + Bu(t) = f(t, u(t)),$$

where A and B are (usually unbounded) linear operators in a Banach space. These problems arise often in the study of partial differential equations. As is usual we control a nonlinear perturbation (possibly involving spatial derivatives) by the linear terms, which contain higher order spatial derivatives. But contrary to the usual method of reducing the problem to a first order system in some "energy" norm space (as in [9, 5]), we use the factoring method of [10]. This method allows the equation to be written as an integral equation containing a double integral involving the nonlinearity, reflecting the fact that the equation is second order.

There are several advantages to this approach which will be illustrated fully in the examples. First, we show that the equation is locally well posed if (loosely) the nonlinearity satisfies the local Lipschitz condition

$$\| f(t, u_1) - f(t, u_2)\| \leqslant C_1 \|u_1 - u_2\|_A + C_2 \|u_1 - u_2\|_{B^{1/2}}.$$

(By absorbing linear terms into f, only the highest order derivatives of the operators A and B need be retained.) For partial differential equations, this gives a good rule of thumb for determining if a certain problem is locally well posed. Energy methods can then be used to show global existence.

Second, this method applies to both hyperbolic and parabolic problems.

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In the case of hyperbolic problems, we usually obtain results equivalent to results derived using "energy spaces" (as in the wave equation, f must be locally Lipschitz continuous with respect to $\|\nabla u\|$). But for parabolic problems we obtain well posedness for a much larger class of nonlinearities than is usually obtained. This will be illustrated in the examples.

Last, using these techniques, it is quite easy to determine if the semigroups controlling the equation are analytic. This is usually easy to determine using product spaces [12, 14], but in certain cases this may be more difficult to do. One case in point is illustrated in a paper by Holmes and Marsden [5]. In this paper they write the equation of motion for a thin panel as a system in the "energy" space $H_0^2 \oplus L^2$. In this space it is unclear if the semigroup controlling the equation is analytic and in fact they conjectured that it was *not analytic*. But in our last example we show using nothing more than L'Hospital's rule that the semigroup *is analytic*.

The first example of our abstract theorem is the Klein–Gordon equation

$$u_{tt} - \alpha\, \Delta u_t + \beta u_t - \Delta u = \mu u - b\, |u|^{q-1}\, u, \qquad \alpha, \beta, b \geq 0, \quad \mu \in \mathbb{R}.$$

It has been shown [2, 9, 13, 14], that if the nonlinearity is locally Lipschitz continuous from the Sobolev space $H_0^{1,2}(\Omega)$ (Ω bounded with smooth boundary $\partial\Omega$ or $\Omega = \mathbb{R}^3$) into $L^2(\Omega)$, then (using Sobolev imbedding theorems and standard energy methods) global solutions exist for $\Omega \subseteq \mathbb{R}^3$ and $1 \leq q \leq 3$. *But we show that for* $\alpha > 0$, *the nonlinearity may be locally Lipschitz continuous from* $H_0^{2,2}(\Omega)$ *to* $L^2(\Omega)$ *which gives us global solutions for* $q \geq 1$ *and* $\Omega \subseteq \mathbb{R}^3$. It would be interesting to study the behavior of these solutions as $\alpha \downarrow 0$.

Our examples also include the dynamical version of the von Karmán equation and the vibrating beam equation. We use our abstract results to give local strong solutions and then construct energy functions to prove the existence of global strong solutions and to study their stability. Using the Center Manifold Theorem [4], we are also able to study bifurcation. It must be stressed that these methods apply to a much larger class of problems than those considered here. Also note that the coefficients of the differential operators need not be constant, and in fact the operators need not even commute.

The abstract approach we use in Section 2 involves factoring linear operators. The operational calculus for self adjoint operators and perturbation results makes this approach quite general. Because the factored operators may be complicated, we use Lipschitz conditions involving simpler operators B_j (often Δ^j, where Δ is the Laplace operator) which are in some sense dominated by the factored operators.

We suggest that many readers may wish to skip Section 2 temporarily (possibly after skimming over hypothesis (H1) and (H2) and Theorem 2.2) and go directly to the examples in Sections 3, 4, and 5.

We finally mention that G. Ponce [8] has independently studied some classes of nonlinear evolution equations one of which is rather similar to the one considered by us in Example 3. The techniques used by Ponce are totally different.

2. The Abstract Results

The results we consider in this section are generalizations of work done in [10]. As such, we assume the reader is familiar with the Spectral Theorem and several function analytic terms such as "Banach space" and "semigroup." In this section we prove existence and uniqueness results for the abstract Cauchy problem

$$u''(t) + Au'(t) + Bu(t) = f(t, u(t)), \qquad u(0) = \phi, \quad u'(0) = \psi \tag{2.1}$$

in a Banach space χ, where A and B are unbounded linear operators. Although in our examples, A and B commute, this is not a necessary condition.

Problem (2.1) is said to be *well-posed locally* if there exists a dense set $D \subset \chi$ such that if $\phi, \psi \in D$, then there exists a $T > 0$ and an unique function u: $[0, T] \to \chi$ which satisfies (2.1).

We assume the existence of (linear) operators A_k, $k = 1, 2$, such that $A_1 + A_2 = -A$, $A_2 A_1 = B$, and A_k generates a (C_0)-semigroup T_k, $k = 1, 2$. (Actually we need that closure $(A_2 A_1) = B$ and closure $(A_1 + A_2) = -A$.) As can be seen from the examples these are reasonable assumptions.

We say that a function u is a mild solution to (2.1) if u is continuous and satisfies

$$u(t) = T_1(t)\,\phi + \int_0^t T_1(t-\tau)\, T_2(\tau)(\psi - A_1\phi)\, d\tau$$

$$+ \int_0^t \int_s^t T_1(t-\tau)\, T_2(\tau - s)\, f(s, u(s))\, d\tau\, ds. \tag{2.2}$$

The motivation for this form of a mild solution came from studying (2.1) in the factored form

$$\left(\frac{d}{dt} - A_1\right)\left(\frac{d}{dt} - A_2\right) u = f$$

and is discussed in Sandefur [10]. But in that paper, the order of integration in the double integral was reversed and the full advantage of this method was not utilized. Note that in (2.2), the inner integral does not depend on the nonlinearity. If we formally compute the first integral, we have

$$u(t)=T_1(t)\,\phi+\int_0^t T_1(t-\tau)\,T_2(\tau)(\psi-A_1\phi)\,d\tau$$

$$+(A^2-4B)^{-1/2}\int_0^t (T_2(t-s)-T_1(t-s))\,f(s,u(s))\,ds.$$

We therefore expected (2.2) to be *well-posed locally* (in the same sense as (2.1)) if f is Lipschitz continuous with respect to $(A^2-4B)^{1/2}$, or equivalently A or $B^{1/2}$, (whichever has the higher order if they are differential operators). This is essentially what we will prove, although it is somewhat hidden in the technical details. If ϕ, ψ are "nice," that is, $\phi, \psi \in \bigcap_{j,k=1}^2 D(A_jA_k)$, and $f(\cdot, u(\cdot)) \in C^2(\mathbb{R}_+, \chi)$ whenever $u(\cdot) \in C^2(\mathbb{R}_+, \chi)$, then *a mild solution is a strong solution.* This is a consequence of the following lemma, which is proved in Kato [6, p. 486].

LEMMA 2.1. *If $g \in C^1(\mathbb{R}_+, \chi)$ and $w(t)=\int_0^t T(t-s)\,g(s)\,ds$, where $T(\cdot)$ is a semigroup generated by a linear operator A on χ, then $w \in C(\mathbb{R}_+, D(A)) \cap C^1(\mathbb{R}_+, \chi)$ and $w'(t)=T(t)\,g(0)+\int_0^t T(t-s)\,g'(s)\,ds$ and $Aw(t)=w'(t)-g(t)$.*

(Above $\mathbb{R}_+=[0,\infty)$ and $D(A)$ is the domain of A).

The details of the use of this lemma to show regularity are identical to those employed by one of the authors in [10, Theorem 4.1] to prove a similar result, even though the class of nonlinearities f, has been expanded considerably. These techniques can also be used to prove standard regularity theorems when A_1 and A_2 are analytic. In Section 3 we shall discuss this matter further.

We solve Eq. (2.2) using successive approximations. Define $f_n(s)=f(s, u_n(s))$ and

$$u_0(t)=T_1(t)\,\phi+\int_0^t T_1(t-\tau)\,T_2(\tau)(\psi-A_1\phi)\,d\tau, \tag{2.3}$$

$$u_{n+1}(t)=u_0(t)+\int_0^t\int_s^t T_1(t-\tau)\,T_2(\tau-s)\,f_n(s)\,d\tau\,ds. \tag{2.4}$$

We will show that, under certain conditions on f, the $u_n(t)$ converge to some function $u(t)$, uniformly for $t \in [0, T)$ for some $T>0$.

One advantage of Eq. (2.2) is the double integral. By integrating with respect to τ we get, in some sense, $(A^2-4B)^{-1/2}$ which may "cancel" some of the nonlinearity f. Because this operator may be difficult to deal with, we hypothesize the existence of a set of closed linear operators B_j, $j = 1, 2, \ldots, m$ with $D = \bigcap_{j=1}^{m} D(B_j)$ dense in χ and which satisfy hypotheses (H1) and (H2). For $m = 1$ and $B_1 = I$, the identity, these hypotheses are the same as in [10]. If in (2.1) A and B are differential operators of order p and q, respectively, then normally $B_j = \Delta^{j/2}$ for $j = 0, 1, \ldots, m$, $m = \max\{2p, q\}$. Here $\Delta^0 \equiv I$ and $\|\Delta^{1/2}u\| = -\langle \Delta u, u\rangle = \|\nabla u\|$, where the last norm is in $L^2(\Omega, \mathbb{C}^n)$. In other words, the B_j are in some sense dominated by $(A^2 - 4B)^{1/2}$. Here it is implied that Δ has certain boundary conditions associated with it. These conditions depend on A and B.

(H1) There exists a positive nondecreasing function $g\colon \mathbb{R}_+ \to (0, \infty)$ and an interval $[0, T]$, $T > 0$, such that

(i) $\|f(t, \phi)\| \leqslant g(K)$ and

(ii) $\|f(t, \phi) - f(t, \psi)\| \leqslant g(K) \sum_{j=1}^{m} \|B_j(\phi - \psi)\|$ when $t \in [0, T]$, $\phi, \psi \in D$ and $\|\phi\|$, $\|\psi\|$, $\|B_j\phi\|$, $\|B_j\psi\| \leqslant K$ for $j = 1, \ldots, m$.

(H2) Let $u_k(t)$, $k = 0, 1, \ldots$, be as defined in (2.3) and (2.4) for $t \in [0, T)$, $T > 0$. Note that u_0 is uniformly bounded on $[0, T)$. Then,

(i) $u_k(t) \in D$ for $k = 0, 1, \ldots$, and $t \in [0, T)$,

(ii) $\|B_j u_{k+1}(t)\| \leqslant \|B_j u_0(t)\| + g(K) \int_0^t \|f_k(s)\|\, ds$, and

(iii) $\|B_j(u_{k+1}(t) - u_k(t))\| \leqslant g(K) \int_0^t \|f_k(s) - f_{k-1}(s)\|\, ds$,

where $t \in [0, T)$, $\|u_k(t)\|$, $\|u_{k-1}(t)\| \leqslant K$ for $t \in [0, T)$, $j = 1, \ldots, m$ and $k = 1, 2, \ldots$.

It will be demonstrated in the examples that these hypotheses are easy to apply.

THEOREM 2.2. *Suppose A_1 and A_2 generate C_0-semigroups T_1, T_2. Also suppose there exists closed linear operators B_j, $j = 1, \ldots, m$ with $\bigcap_{j=1}^{m} D(B_j) \cap D(A_1) \cap D(A_2)$ dense in χ and such that* (H1) *and* (H2) *are satisfied. Then* (2.2) *is well-posed locally.*

Proof. The proof consists in showing that the functions $\{u_n\}$ defined in (2.3) and (2.4) converge uniformly to a function u for $t \in [0, T)$ for some $T > 0$.

First we show, using induction, that for some $T > 0$ and $K > 0$ that $\|u_k(t)\|$, $\|B_j u_k(t)\| \leqslant K$ for $j = 1, \ldots, m$, $k = 1, 2, \ldots$, and $t \in [0, T)$. For simplicity we assume $m = 1$.

It is easy to show that for some $K > 0$ and $T > 0$, that

$$\|u_k(t)\|, \|B_1 u_k(t)\| \leqslant K/2, \qquad t \in [0, T) \tag{2.5}$$

for $k=0$. Assume (2.5) for $k=0,\ldots, n$ for some n and with $K/2$ replaced with K. Then

$$\|u_{n+1}(t)\| \leqslant \|u_0(t)\| + \left\| \int_0^t \int_s^t T_1(t-\tau)\, T_2(\tau - s)\, f_n(s)\, d\tau\, ds \right\|$$

$$\leqslant K/2 + C \int_0^t (t-s)\, \|f_n(s)\|\, ds \leqslant K/2 + Cg(K)\, t^2/2 \leqslant K$$

for $t \in [0, (K/Cg(K))^{1/2}]$ for some $C>0$. Also

$$\|B_1 u_{n+1}(t)\| \leqslant \|B_1 u_0(t)\| + g(K) \int_0^t \|f_n(s)\|\, ds$$

$$\leqslant K/2 + g^2(K)\, t \leqslant K$$

for $t \in [0, K/g^2(K)]$. So let $T = \min\{K/g^2(K), (K/Cg(K))^{1/2}\}$. Thus (2.5) holds (with $K/2$ replaced with K) for $k=0, 1,\ldots$.

It is also easy to show that

$$\|u_{k+1}(t) - u_k(t)\|,\ \|B_1(u_{k+1}(t) - u_k(t))\| \leqslant (Mt)^k/k! \tag{2.6}$$

for $t \in [0, T)$ and $k=1$. Assume (2.6) holds for $k=1,\ldots, n$ for some n. Then

$$\|u_{n+1}(t) - u_n(t)\| = \left\| \int_0^t \int_s^t T_1(t-\tau)\, T_2(\tau-s)(f_n(s) - f_{n-1}(s))\, d\tau\, ds \right\|$$

$$\leqslant g(K) \int_0^t \int_s^t \|B_1(u_n(s) - u_{n-1}(s))\|\, d\tau\, ds$$

$$\leqslant g(K) \int_0^t \int_s^t (Ms)^n/n!\, d\tau\, ds$$

$$= M^n g(K)\, t^{n+2}/(n+2)! \leqslant (Mt)^{n+1}/(n+1)!$$

when $M \geqslant Tg(K)/2$. Similarly for $\|B_1(u_{n+1}(t) - u_n(t))\|$ with $M \geqslant g(K)$. Therefore, (2.6) holds for all k with $M = (1 + T/2)\, g(K)$. Thus, u_n converges uniformly on $[0, T)$ to a solution u of (2.2).

The uniqueness follows by an application of Gronwall's inequality as in [10]. Q.E.D.

We now discuss global continuation of solutions.

COROLLARY 2.3. *Let the assumptions of Theorem* 2.2 *hold and let u be the solution to* (2.2) *on* $[0, T)$. *Let* $\|u\|$ *and* $\|B_j u\|$, $j=1,\ldots, m$, *be uniformly* *bounded on* $[0, T)$ *and* $T<\infty$. *Then* $\lim_{t\uparrow T} u(t) \equiv u(T)$ *exists. Define for* $t>T$,

$$u_0(t) = T_1(t)\,\phi + \int_0^t T_1(t-\tau)\,T_2(\tau)(\psi - A_1\phi)\,d\tau$$
$$+ \int_0^T \int_s^t T_1(t-\tau)\,T_2(\tau - s)\,f(s, u(s))\,d\tau\,ds. \tag{2.7}$$

Assume that for each $K>0$ *there is a nondecreasing function* g_K *such that* (H1) *and* (H2) *hold with* f *replaced by* f_K *(defined by* $f_K(t, \phi) = f(t+K, \phi)$*) and* u_0 *as in* (2.7). *Then* u *can be extended to a solution of* (2.2) *on the interval* $[0, T+\varepsilon)$ *for some* $\varepsilon>0$.

The proof consists of applying the same techniques as before to functions $v_n(t) \equiv u_n(t+T)$. See [10] for more details.

Let $B_1 = I$, $B_2 = A_1$, and $m=2$. Then local existence can also be shown with (H2) replaced by

(H3) Let $\phi_j \in C^1([0, T], \chi) \cap C([0, T], D(A_1))$ for $j = 1, 2$. Then

$$\left\| \frac{d}{dt} f(t, \phi_1(t)) \right\| \leqslant g(K)$$

and

$$\left\| \frac{d}{dt} (f(t, \phi_1(t)) - f(t, \phi_2(t))) \right\| \leqslant g(K)[\,\|\phi_1(t) - \phi_2(t)\|$$
$$+ \|A_1(\phi_1(t) - \phi_2(t))\| + \|\phi_1'(t) - \phi_2'(t)\|\,]$$

for $t \in [0, T]$, g as in (H1), and where

$$\sup_{\substack{t\in[0,T]\\ j=1,2}} \{\|\phi_j(t)\|, \|\phi_j'(t)\|, \|A_1\phi_j(t)\|\} \leqslant K.$$

For the wave equation, Reed [9] has another hypothesis that could also replace (H2). The disadvantage of these alternatives is that they do not take advantage of the strong connection that usually exists between the semigroups and the nonlinearities and especially the fact that the equation is second order.

We also note that f could also be a function of u_t. Letting $f_k(t) = f(t, u_k(t), u_{k,t}(t))$, (H1) and (H2) would involve inequalities of the form

$$\|f(t, u, u_t) - f(t, v, v_t)\| \leqslant g(K)\left(\sum_{j=1}^{m} \|B_j(u-v)\| + \|u_t - v_t\|\right)$$

and

$$\|u_{k+1,t}(t)-u_{k,t}(t)\| \leqslant g(K)\int_0^t \|f_k(s)-f_{k-1}(s)\|\,ds.$$

An example of this is given in our last section.

We now give an approximation result

COROLLARY 2.4. *Let the hypotheses of Theorem 2.2 hold for the nonlinear perturbations f and f_n, $n=1, 2,\ldots$, with the same g in each case. Assume*

$$\|B_j(u(t)-u_n(t))\| \leqslant g(K)\int_0^t \|f(s, u(s))-f_n(s, u_n(s))\|\,ds \tag{2.8}$$

when $\|u(t)\|$, $\|u_n(t)\| \leqslant K$, $t\in[0, T)$, $j=1,\ldots, m$, and where u_n is the solution of (2.2) with f replaced by f_n. Assume that for every $\varepsilon>0$ and $K>0$ there is an $N>0$ such that $\|f_n(t,\phi)-f(t,\phi)\| \leqslant \varepsilon$ when $\|B_j(\phi)\| \leqslant K$ for $j=1,\ldots, m$, $t\in[0, T)$ and $n\geqslant N$. Then $u_n(t)\to u(t)$ locally uniformly for $t\in[0, T_0]$, for some $T_0>0$.

Proof. We note that

$$\begin{aligned}
\|u(t)-u_n(t)\| &= \left\| \int_0^t\int_s^t T_1(t-\tau)\,T_2(\tau-s)(f(s, u(s))-f_n(s, u_n(s)))\,d\tau\,ds \right\| \\
&\leqslant M\int_0^t\int_s^t \|f(s, u(s))-f_n(s, u_n(s))\|\,d\tau\,ds \\
&\leqslant M\int_0^t\int_s^t \|f(s, u(s))-f(s, u_n(s))\| \\
&\qquad + \|f(s, u_n(s))-f_n(s, u_n(s))\|\,d\tau\,ds.
\end{aligned}$$

For $0<T_0$ small enough, u and u_n are uniformly bounded by K for $t\in[0, T_0]$. Therefore for any $\varepsilon>0$, there is an $N>0$ such that for $n>N$,

$$\|u(t)-u_n(t)\| \leqslant M\int_0^t\int_s^t g(K)\sum_{j=1}^m \|B_j(u(s)-u_n(s))\|+\varepsilon\,d\tau\,ds.$$

By (2.8)

$$\begin{aligned}
\|B_j(u(t)-u_n(t))\| &\leqslant g(K)\int_0^t \|f(s, u(s))-f_n(s, u_n(s))\|\,ds \\
&\leqslant g^2(K)\int_0^t \sum_{j=1}^m \|B_j(u(s)-u_n(s))\|+\varepsilon\,ds
\end{aligned}$$

as above. Let $\phi_n(t) = \|u(t) - u_n(t)\| + \sum_{j=1}^{m} \|B_j(u(t) - u_n(t))\|$. We have

$$\phi_n(t) \leqslant C \int_0^t \phi_n(s)\, ds + \varepsilon_n$$

for some $C > 0$ and $\varepsilon_n \to 0$. By Gronwall's inequality, $u_n(t) \to u(t)$ in χ as $n \to \infty$, uniformly for $t \in [0, T_0]$, T_0 some positive constant.

3. The Klein–Gordon Equation

Here we shall prove existence of global strong solutions and study bifurcations for the strongly damped Klein–Gordon equation

$$u_{tt} - 2\alpha\, \Delta u_t - \Delta u = \mu u - b\, |u|^{(q-1)}\, u, \tag{3.1}$$

with $\mu \in \mathbb{R}$, $\alpha > 0$, $q \geqslant 1$ on a bounded domain $\Omega \subseteq \mathbb{R}^n$, $n = 1, 2, 3$, with $\partial\Omega$ smooth and boundary condition $u(t, x) = 0$ for $x \in \partial\Omega$. (Using spectral decomposition instead of eigenfunction expansion we can derive local existence for $\Omega = \mathbb{R}^3$. Global existence and energy arguments are identical.)

As is well known, if $\alpha = 0$, $\mu = 0$ and $1 \leqslant q \leqslant 3$, then (3.1) has global *strong* solutions. (This is also what is obtained using the results of Section 2.) If $q > 3$, then there are *weak* global solutions. (See [9].) Our results show that if $\alpha > 0$, then there are also global strong solutions when $q > 3$, which leads to the following interesting questions: Fix the initial conditions and $q > 3$. Denote by u^α the corresponding solution of (3.1) and by v a corresponding weak solution of (3.1) when $\alpha = 0$.

Open Problems. Does $\lim_{\alpha \to 0} u^\alpha$ exists in some strong sense? What is the relation between u^α and v? (For a discussion of these types of problems, see [7].)

We pose (3.1) in the complex Hilbert space $L^2(\Omega)$ and by Δ we mean the closure of the Laplacian on the set $\{u \in C^2(\Omega)\colon u = 0 \text{ on } \partial\Omega\}$ with respect to the L^2-norm. $-\Delta$ has eigenvalues λ_j, $j = 1, 2, \ldots$, $\lambda_j > 0$ such that $\lim_{j \to \infty} \lambda_j = \infty$ and a corresponding complete set of orthonormal eigenvectors e_j (cf. [3, p. 1335]).

Let $A_k = \text{closure } (\alpha\Delta + (-1)^k(\alpha^2\Delta^2 + \Delta)^{1/2})$ for $k = 1, 2$. Then, it is seen that A_k is the sum of a nonpositive self-adjoint operator and a bounded operator (which corresponds to the finite number of eigenvalues for which $(\alpha^2\lambda_j^2 - \lambda_j)$ is negative). Therefore, A_k generates an analytic semigroup T_k, $k = 1, 2$.

Now, we let $B_1 = I$, the identity and $B_2 = \Delta$. To apply Theorem 2.2 it suffices to show that

(i) $\|f(u)\| \leqslant g(K)$, $\|f(u)-f(v)\| \leqslant g(K)(\|\Delta(u-v)\| + \|u-v\|)$,

(ii) $\|\Delta u_{n+1}\| \leqslant C(\|\Delta u_0\| + \int_0^t \|f_n(s)\|\, ds)$, and

(iii) $(\|\Delta(u_{n+1}-u_n)\| \leqslant C\int_0^t \|f_n(s)-f_{n-1}(s)\|\, ds$

when $\|u\|$, $\|v\|$, $\|\Delta u\|$, $\|\Delta v\| \leqslant K$, g is as in (H1), C is some positive constant independent of n and $f_n(s) \equiv f(u_n(s)) = \mu u_n - b\,|u_n|^{(q-1)}\,u_n$.

Let $\|u\|_\infty = \sup_{x\in\Omega} |u(x)|$ and $\|u\|_p = (\int_\Omega |u|^p\, dx)^{1/p}$, $p \geqslant 1$. Then, from the Sobolev imbeddings, $\|u\|_\infty \leqslant C\,\|\Delta u\|$ and $\|u\|_p \leqslant C\,\|\Delta u\|$ for some $C>0$. Thus, (i) follows immediately.

Since (ii) and (iii) are similar we shall only verify (iii). Let $g(s) = f_n(s) - f_{n-1}(s)$. Then

$$\|\Delta(u_{n+1}-u_n)\| = \left\| \Delta \int_0^t \int_s^t \exp((t-\tau)\,A_1)\exp((\tau-s)\,A_2)\,g(s)\,d\tau\,ds \right\|$$

$$= \left\| \sum_j (-\lambda_j) \int_0^t \int_s^t \exp((t-\tau)(-\alpha\lambda_j-\beta_j)) \right.$$

$$\left. \times \exp((\tau-s)(-\alpha\lambda_j+\beta_j))\,g_j(s)\,d\tau\,ds \right\|$$

by the operational calculus for self-adjoint operators, with $\beta_j = (\alpha^2\lambda_j^2-\lambda_j)^{1/2}$ and $g_j(s) = \langle g(s), e_j \rangle$.

$$\|\Delta(u_{n+1}-u_n)\|^2 = \left\| \sum \frac{-\lambda_j}{2\beta_j} \int_0^t (\exp((t-s)(-\alpha\lambda_j+\beta_j)) \right.$$

$$\left. -\exp((t-s)(-\alpha\lambda_j-\beta_j)))\,g_j(s)\,e_j\,ds \right\|^2$$

$$\leqslant C \left\| \int_0^t (T_2(t-s)-T_1(t-s))\,g(s)\,ds \right\|^2$$

where $C = \sup_j |\lambda_j/(\alpha^2\lambda_j^2-\lambda_j)^{1/2}|$. (We can treat the finite number of eigenvalues for which $\beta_j = 0$ separately to arrive at the same inequality.) Thus, $\|\Delta(u_{n+1}(t)-u_n(t))\| \leqslant C\int_0^t \|g(s)\|\, ds$. Hence, we have the existence of mild local solutions to (3.1).

The existence of global *real* solutions with $b>0$ follows by considering the energies

$$E_1(t) = (\|\nabla u\|^2 + \|u_t\|^2 - \mu\,\|u\|^2)/2 + b\,\|u^{(q+1)/2}\|^2/(q+1)$$

and

$$E_2(t) = [\alpha\,\|\Delta u\|^2 + \|u_t\|^2 + \|\nabla u\|^2 - \mu\,\|u\|^2$$
$$+ 2b\,\|u^{(q+1)/2}\|^2/(q+1)]/2\alpha - \langle u_t, \Delta u \rangle,$$

where $\langle \cdot, \cdot \rangle$ means inner product. From (3.1) and integration by parts we have

$$\frac{dE_1(t)}{dt} = -2\alpha \|\nabla u_t\|^2,$$

which implies that $E_1(t)$ is nonincreasing. There is no problem if $\mu \leqslant 0$, so assume μ is positive. Now, suppose $\|u\|^2$ is unbounded. Then, there exists an increasing sequence t_n such that $\|u(t_n)\|^2 > n$, $n = 1, 2, \ldots$. Let $B_n = \{x \in \Omega: |u(t_n)|^{q-1} > (q+1)(\mu+1)/2b\}$. B_n has positive measure and $\int_{B_n} u^2(t_n)\,dx \to \infty$ as $n \to \infty$. So

$$\begin{aligned} E_1(t_n) &\geqslant \int_\Omega b\frac{|u(t_n)|^{(q+1)}}{(q+1)} - \frac{\mu}{2}|u(t_n)|^2\,dx \\ &= \int_\Omega |u(t_n)|^2(2b\,|u(t_n)|^{q-1}/(q+1) - \mu)/2\,dx \\ &\geqslant \int_{B_n} u^2(t_n)/2\,dx - C, \end{aligned}$$

with C some fixed positive constant. Since $\int_{B_n} u^2(t_n)\,dx \to \infty$, $E_1(t_n) \to \infty$ which is a contradiction and therefore $\|u\|$ is a bounded function in t. From the definition of E_1 we then see that $\|\nabla u\|$, $\|u_t\|$ and $\|u^{(q+1)/2}\|$ are bounded uniformly in t.

We now consider E_2.

$$\begin{aligned} \frac{dE_2(t)}{dt} = \int_\Omega \Big\{ &(\Delta u)\,\alpha\,\Delta u_t + u_t u_{tt}/\alpha + \nabla u_t \cdot \nabla u_t \\ &- (\Delta u)\,u_{tt} + \nabla u \cdot \nabla u_t/\alpha + \frac{b}{\alpha}|u|^q u_t \operatorname{sign} u + \mu u u_t/\alpha \Big\}\,dt. \end{aligned}$$

Since $u_{tt} = \alpha\,\Delta u_t + \Delta u - b\,|u|^{q-1}u + \mu u$, $\alpha\,\Delta u_t = u_{tt} - \Delta u + b\,|u|^{q-1}u - \mu u$, and $\nabla|u| = \operatorname{sign} u\,\nabla u$ we get

$$\begin{aligned} \frac{dE_2(t)}{dt} &= \int_\Omega -(\Delta u)^2 + b\,|u|^q \operatorname{sign} u\,\Delta u + \mu\,\nabla u \cdot \nabla u\,dx \\ &= -\int_\Omega (\Delta u)^2 + (b\,|u|^{q-1} - \mu)\,|\nabla u|^2\,dx. \end{aligned}$$

Since $\|\nabla u\|$ is bounded,

$$\frac{dE_2}{dt}(t) \leqslant C$$

for some constant $C \geqslant 0$. Thus $E_2(t) \leqslant Ct + C_1$, with C_1 some positive constant. Using the inequality

$$2\,|\langle \Delta u, u_t \rangle| \leqslant \beta\,\|\Delta u\|^2 + \|u_t\|^2/\beta,$$

with β arbitrarily small and since $\|u\|$ and $\|u_t\|$ are bounded, we have from E_2 that $\|\Delta u\|$ is bounded on finite intervals. Therefore, from Corollary 2.3 we have global existence of solutions of (3.1) *for all* $q \geqslant 1$.

To show that solutions are strong solutions it is necessary to prove that u_{tt}, Δu_t, and Δu are in $L^2(\Omega)$. We show that Δu_t is, the others being similar. Using Lemma 2.1 on

$$u(t) = u_0(t) + \int_0^t \int_s^t T_1(t-\tau)\, T_2(\tau - s)\, f(u(s))\, d\tau\, ds$$

(and assuming the initial data are "nice" enough so that Δu_0, Δu_{0t}, and u_{0tt} exist) we have after a substitution and a change in the order of integration that

$$\begin{aligned}\Delta u_t(t) &= \Delta u_{0t} + \Delta \frac{d}{dt} \int_0^t T_2(t-\tau) \int_0^\tau T_1(\tau - s)\, f(u(s))\, ds\, d\tau \\ &= u_{0t} + \Delta \int_0^t T_1(t-s)\, f(u(s))\, ds \\ &\quad + A_2 \int_0^t T_2(t-\tau)\, \Delta \int_0^\tau T_1(\tau - s)\, f(u(s))\, ds\, d\tau.\end{aligned}$$

Since (i) A_2 is bounded (its spectrum is bounded), (ii) A_1, A_2, Δ, T_1, and T_2 all commute, (iii) $D(\Delta) = D(A_1)$ by the operational calculus of self-adjoint operators and

(iv) $\int_0^t T_j(t-s)\, f(s)\, ds \in D(A_j)$ if $f \in C(\mathbb{R}_+, \chi)$, $j = 1, 2$,

we have that each of the above terms is in $L^2(\Omega)$.

We now study bifurcation by letting μ be a parameter and using the Center Manifold Theorem [4]. To do this we need to reduce (3.1) to a first order system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + F(u, v), \tag{3.2}$$

where $F(u, v) = (v, f(u))^{\mathrm{T}}$, T is the transpose operator. It is easy to see that a strong solution u to (3.1) gives a strong solution to (3.2) with $v = u_t - A_1 u$. Therefore we have existence of global strong solutions of (3.2).

Note that by the operational calculus for self-adjoint operators [3], and since $2\alpha\lambda_j \geqslant \alpha\lambda_j + (\alpha^2\lambda_j^2 - \lambda_j)^{1/2} > \alpha\lambda_j$ for large j, we have $D(\Delta^\gamma) = D(A_1^\gamma)$ for $0 \leqslant \gamma \leqslant 1$.

Let $F((\phi, \psi)^{\mathrm{T}}) = (\psi, \mu\phi - b\,|\phi|^{q-1}\,\phi)^{\mathrm{T}}$. F has a Fréchet derivative F' from $D(A_1^\gamma) \oplus L^2(\Omega)$ into $L^2(\Omega) \oplus L^2(\Omega)$ which is continuous, and $F'(0) = \left(\begin{smallmatrix} 0 & 1 \\ \mu & 0 \end{smallmatrix}\right)$. (The norm on $D(A_1^\gamma)$ is $\|\phi\|_\gamma^2 = \|\phi\|^2 + \|A_1^\gamma \phi\|^2$.) Let $\mu = \lambda_1 + 2\alpha\lambda_1 \varepsilon$ and rewrite (3.2) as

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} &= A\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 2\alpha\lambda_1 u \end{pmatrix} + G\begin{pmatrix} u \\ v \end{pmatrix}, \\ \varepsilon' &= 0, \end{aligned} \tag{3.3}$$

where $A = \left(\begin{smallmatrix} A_1 & 1 \\ \lambda_1 & A_2 \end{smallmatrix}\right)$ and $G\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ -b|u|^{q-1}u \end{smallmatrix}\right)$.

If $(\phi, \psi)^{\mathrm{T}} = \sum (a_j e_j, b_j e_j)^{\mathrm{T}} = \sum (a_j, b_j)^{\mathrm{T}}\, e_j$, then

$$A\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum \begin{pmatrix} -\alpha\lambda_j - (\alpha^2\lambda_j^2 - \lambda_j)^{1/2} & 1 \\ \lambda_1 & -\alpha\lambda_j + (\alpha^2\lambda_j^2 - \lambda_j)^{1/2} \end{pmatrix} \begin{pmatrix} a_j \\ b_j \end{pmatrix} e_j$$

and $\sigma(A) = \{\beta_j^\pm = -\alpha\lambda_j \pm (\alpha^2\lambda_j^2 - \lambda_j + \lambda_1)^{1/2}; j = 1,...\}$. Note that $\beta_1^+ = 0$ and $\operatorname{Re}\{\beta_1^-, \beta_{j, j=2,...}^\pm\} \leqslant -\delta < 0$ for some $\delta > 0$. The eigenvectors associated with $\beta_1^\pm$ are $f^\pm = (1, \pm\alpha\lambda_1 + (\alpha^2\lambda_1^2 - \lambda_1)^{1/2})^{\mathrm{T}}\, e_1$. The eigenvectors associated with $\beta_j^\pm$, $j = 2,...,$ are orthogonal to $f^\pm$.

Let P be the projection onto f^+ along f^-, i.e.,

$$P(a, b)^{\mathrm{T}}\, e_1 = \langle (\alpha\lambda_1 - (\alpha^2\lambda_1^2 - \lambda_1)^{1/2}, 1)^{\mathrm{T}}, (a, b)^{\mathrm{T}} \rangle\, f^+/2\alpha\lambda_1.$$

Equation (3.3) can then be rewritten as

$$\begin{aligned} s'(t)\, f^+ &= s(t)\, \varepsilon f^+ + P(0, -b\,|u|^q\, u)^{\mathrm{T}}, \\ y'(t) &= By(t) + (I - P)(0, -b\,|u|^q\, u)^{\mathrm{T}}, \\ \varepsilon' &= 0, \end{aligned} \tag{3.4}$$

where y is in the space $H = \{f^-\} \oplus \{f^+, f^-\}^\perp$, with $\{f^+, f^-\}^\perp$ the orthogonal complement of f^+ and f^- in $L^2(\Omega)$. B is a linear operator on H with $\operatorname{Re}\sigma(B) \leqslant -\delta$. By Theorem 6.2.1 of Henry [4], there is a Lipschitzian local invariant manifold $S = \{y = h(s, \varepsilon);\ |s|, |\varepsilon| < \varepsilon_0\}$ for some $\varepsilon_0 > 0$, tangent to f^+ at the origin. The flow in S may be represented by the ordinary differential equation

$$\begin{aligned} s' f^+ &= s\varepsilon f^+ + PG(sf^+ + h(s, \varepsilon)), \\ \varepsilon' &= 0. \end{aligned}$$

By Corollary 6.2 of Henry [4], $\|h(s, \varepsilon)\| = O(s^2+\varepsilon^2)$ as $(s, \varepsilon) \to (0, 0)$. Therefore $PG(sf^+ + h(s, \varepsilon)) = -C\,|s|^{q-1}\,sf^+ + O(s^{q+1}+\varepsilon s^q)$ as $(s, \varepsilon) \to (0, 0)$ where $C = b\int_\Omega |e_1|^{q-1}\,e_1^2\,dx/2\alpha\lambda_1 > 0$. So the flow on the manifold S may be represented by the solution to

$$\begin{aligned} s' &= \varepsilon s - C\,|s|^{q-1}\,s + O(s^{q+1}, \varepsilon s^q) \\ \varepsilon' &= 0. \end{aligned} \tag{3.5}$$

If $\varepsilon < 0$, the solution to (3.5) decays exponentially and thus the origin is asymptotically stable in Eq. (3.2). Likewise when $\varepsilon = 0$, the origin is stable.

Suppose $\varepsilon > 0$ and small. Since (for ε small and fixed),

$$\varepsilon s - C\,|s|^{q-1}\,s + O(s^{q+1}, \varepsilon s^q) = 0 \tag{3.6}$$

has a positive and negative root (close to zero), the unstable manifold of the origin consists of two stable orbits connecting two fixed points to the origin (see Fig. 1).

We finally remark that analogous bifurcation results for (3.1) were derived by Webb [15] when $\Omega = (0, \pi)$.

4. The Dynamical Von Karmán Equation

Consider

$$\begin{aligned} u_{tt} - \alpha\,\Delta u_t + \Delta^2 u &= [\phi, u] - \lambda u_{x_1 x_1}, \\ \Delta^2 \phi &= -\tfrac{1}{2}[u, u], \end{aligned} \tag{4.1}$$

for $(t, x) \in \mathbb{R}_+ \times \Omega$, $\lambda \in \mathbb{R}$, $\alpha \geqslant 0$ and with boundary conditions

$$u = \Delta u = 0, \qquad (\phi)_N = (\Delta\phi)_N = 0$$

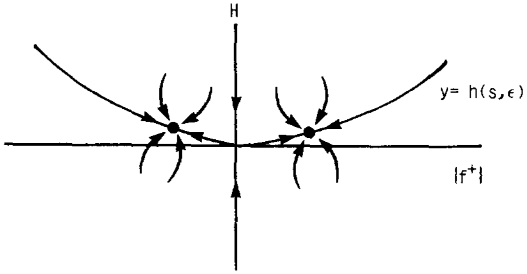

Figure 1

for $x \in \partial\Omega$, and where N denotes the outward normal. Here $\Omega \subseteq \mathbb{R}^2$ is a bounded region with smooth boundary $\partial\Omega$ and

$$[u, v] = u_{x_1x_1} v_{x_2x_2} - 2u_{x_1x_2} v_{x_1x_2} + u_{x_2x_2} v_{x_1x_1}. \tag{4.2}$$

For the physical relevance of the Von Karmán equation, see [1, 11].

Define

$$\Delta^{-1} f(y) = \int_{\mathbb{R}^2} G(x, y) f(x)\, dx$$

for $f \in C_0^1(\Omega)$, and where $G(x, y)$ is the Green function of Δ in Ω with pole at y, i.e., $\Delta G(x, y) = \delta(x-y)$ and $G(x, y) = 0$ if $x \in \partial\Omega$. Then, assuming $\phi \in H_0^{4,2}(\Omega)$, (4.1) is equivalent to

$$u_{tt} - \alpha\, \Delta u_t + \Delta^2 u = -\tfrac{1}{2}[\Delta^{-2}[u, u], u] - \lambda u_{x_1x_1}. \tag{4.3}$$

Here, as in Section 3 (closure of) $-\Delta$ is a positive self-adjoint operator with its first eigenvalue $\lambda_1 > 0$. Note also that Δ^{-1} is not the inverse of this Laplacian, but an inverse of the Laplacian with Neumann boundary conditions. We again have that if the initial data is sufficiently smooth, say $u(0, \cdot) \in H_0^{4,2}(\Omega)$ and $u_t(0, \cdot) \in H_0^{2,2}(\Omega)$, then mild solutions are strong solutions. The proof of this fact is exactly as in Section 3.

We shall now show that (4.3) satisfies the conditions for local existence of solutions. For simplicity we denote $\|u\|_2 = \|u\| + \|\Delta u\|$, where $\|\cdot\|$ is the usual L_2-norm. We now verify (H1)(i) with $B_1 = I$ and $B_2 = \Delta$. We therefore need to find a nondecreasing function $g\colon \mathbb{R}_+ \to (0, \infty)$ such that

$$\int_\Omega |[\Delta^{-2}[u, u], u] - \lambda u_{x_1x_1}|^2\, dx \leqslant g(K) \tag{4.4}$$

when $\|u\|$, $\|\Delta u\| \leqslant K$. Clearly it suffices to obtain this for the case when $\lambda = 0$.

From (4.2) we have

$$\int_\Omega |[\Delta^{-2}[u, u], u]|\, dx \leqslant \max\{a, b, c\}\ \|u\|_2$$

with

$$a = \sup |\Delta^{-2}[u, u]_{x_1x_1}|, \qquad b = \sup 2\, |\Delta^{-2}[u, u]_{x_1x_2}|,$$

and

$$c = \sup |\Delta^{-2}[u, u]_{x_2x_2}|,$$

where sup is taken over Ω. From Sobolev imbeddings and the definition of Δ^{-1},

$$a \leqslant C\,\|\Delta^{-1}[u,u]_{x_1x_1}\| \leqslant C\,\|\Delta^{-1}[u,u]\|_2 \leqslant C\,\|[u,u]\| \leqslant C\,\|\Delta u\|$$

with C denoting several positive constants. Similarly, we get estimates for b and c. Hence (4.4).

We will now verify (H1)(ii). It is obvious that it suffices to consider

$$\begin{aligned}\operatorname{dif}(u,v) &\equiv \|[\Delta^{-2}[u,u],u]-[\Delta^{-2}[v,v],v]\|\\ &\leqslant \|[\Delta^{-2}[u,u],u-v]\| + \|[\Delta^{-2}[u-v,u],v]\|\\ &\quad + \|[\Delta^{-2}[u-v,v],v]\|.\end{aligned}$$

Since $\|\Delta\phi\|$ and $\|\phi\|_2$ are equivalent norms on $H_0^{2,2}(\Omega)$, we can use the previous procedure to obtain

$$\operatorname{dif}(u,v) \leqslant g(K)\,\|u-v\|_2 \tag{4.5}$$

with g as above. Therefore, (H1) is satisfied.

In this example $A_k=(\alpha+(-1)^k\beta)\Delta/2$ where $\beta=(\alpha^2-4)^{1/2}$ and therefore it generates a semigroup $T_k(t)=\exp(A_k t)$ for $k=1,2$. Note that these semigroups are analytic with region of analyticity $\{z: |\arctan(\operatorname{Im} z/\operatorname{Re} z)| < (\pi/2) - a \arctan|\beta/\alpha|\}$, where $a=1$ if β is imaginary and $a=0$ if β is real.

We prove local existence for the cases $\alpha>2$, $\alpha=2$, and $0\leqslant\alpha<2$. We use the fact that

$$\Delta\int_s^t \exp(\tau\Delta)\,\phi\,d\tau = (\exp(t\Delta)-\exp(s\Delta))\,\phi$$

for $\phi\in L_2(\Omega)$.

Case 1. $\alpha>2$. If $n>0$ we have

$$\begin{aligned}&\|\Delta(u_{n+1}(t)-u_n(t))\|\\ &\quad=\left\|\int_0^t \Delta\int_s^t T_1(t-\tau)\,T_2(\tau-s)(f_n(s)-f_{n-1}(s))\,d\tau\,ds\right\|.\end{aligned}$$

A consequence of the operational calculus for self-adjoint operators gives us that

$$\begin{aligned}&\|\Delta(u_{n+1}(t)-u_n(t))\|\\ &\quad=\left\|\int_0^t \Delta\int_s^t \exp((\alpha(t-s)-\beta(t+s)+2\beta\tau)\,\Delta/2)(f_n(s)-f_{n-1}(s))\,d\tau\,ds\right\|\end{aligned}$$

$$= \left\| \int_0^t \beta^{-1}[\exp((\alpha+\beta)(t-s)\,\Delta/2) \right.$$

$$\left. - \exp((\alpha-\beta)(t-s)\,\Delta/2)](f_n(s) - f_{n-1}(s))\,ds \right\|$$

$$= \left\| \beta^{-1} \int_0^t [T_2(t-s) - T_1(t-s)](f_n(s) - f_{n-1}(s))\,ds \right\|$$

$$\leqslant 2\beta^{-1} \int_0^t \|f_n(s) - f_{n-1}(s)\|\,ds.$$

In a similar way we can get bounds on $\|\Delta u_n(t)\|$ thereby establishing (H2).

Case 2. $\alpha = 2$. The previous method does not apply because $\beta = 0$. In this case $T_1(t) = T_2(t)$ and

$$\|\Delta(u_{n+1}(t) - u_n(t))\| = \left\| \Delta \int_0^t \int_s^t T_1(t-s)(f(u_n(s)) - f(u_{n-1}(s)))\,d\tau\,ds \right\|.$$

Since T_1 is analytic, it is known [6] that

$$\|\Delta \exp((t-s)\,\Delta)\| \leqslant C/(t-s)$$

for some $C > 0$. So

$$\|\Delta(u_{n+1}(t) - u_n(t))\| = \left\| \Delta \int_0^t (t-s)\,T_1(t-s)(f_n(s) - f_{n-1}(s))\,ds \right\|$$

$$\leqslant C \int_0^t \|f_n(s) - f_{n-1}(s)\|\,ds.$$

Again we can use similar methods to get bounds on $\|\Delta u_n(t)\|$ showing that (H2) is satisfied.

Case 3. $0 \leqslant \alpha < 2$. Here β is purely imaginary, but the method in Case 1 still works.

Thus for $\alpha \geqslant 0$ we have local existence and uniqueness of solutions of the dynamical Von Karmán equation.

To prove the global existence of *real*-valued solutions we consider the energy function

$$E_1(t) = \int_\Omega \tfrac{1}{2}(|u_t|^2 + |\Delta u|^2 + \Delta^{-1}[u, u]^2/4 + 2\gamma u u_t - \alpha\gamma u\,\Delta u)\,dx$$

with γ an arbitrarily small number to be chosen later. Then, differentiating and integrating by parts,

$$dE_1(t)/dt = \int_\Omega (u_t u_{tt} + u_t \Delta^2 u + \Delta^{-1}[u, u]\, \Delta^{-1}[u_t, u]/2$$
$$+ \gamma u_t^2 + \gamma u u_{tt} - \alpha\gamma u\, \Delta u_t)\, dx.$$

Using the self-adjointness of Δ^{-1} and the identity

$$\int_\Omega v[u, w]\, dx = \int_\Omega [v, u]\, w\, dx$$

for $u, v, w \in H_0^{2,2}(\Omega)$, and letting $v = \Delta^{-2}[u, u]$, we have

$$dE_1(t)/dt = \int_\Omega (u_t u_{tt} + u_t \Delta^2 u + [\Delta^{-2}[u, u], u]\, u_t/2$$
$$+ \gamma u_t^2 + \gamma u u_{tt} - \alpha\gamma u\, \Delta u_t)\, dx.$$

Using the fact that u_{tt} satisfies (4.3) and integrating by parts, we get

$$dE_1(t)/dt = -\alpha \|\nabla u_t\|^2 - \gamma \|u_t\|^2 - \gamma \|\Delta u\|^2 - \tfrac{1}{2} \|\Delta^{-1}[u, u]\|^2$$
$$+ \lambda\gamma \|u_{x_1}\|^2 - \int_\Omega \lambda u_t u_{x_1 x_1}\, dx.$$

Using the inequality $|\langle u, v\rangle| \leqslant \frac{1}{2}(\|u\|^2 + \|v\|^2)$, the Poincaré inequality ($\|u\|^2 \leqslant \lambda_1^{-1} | \|\Delta u\|^2$ for $u \in H_0^{2,2}(\Omega)$ and where λ_1 is the first eigenvalue of $-\Delta$), and choosing γ small enough, we get

$$E_1(t) \geqslant C_1(\|u_t\|^2 + \|\Delta u\|^2) \geqslant C_2\, dE_1(t)/dt \tag{4.6}$$

for some positive constants C_1 and C_2. Hence, for some M, $C > 0$, $E_1(t) \leqslant M \exp(Ct)$. So, using (4.6), we get that $\|u_t\|^2$ and $\|\Delta u\|^2$ are uniformly bounded if $t < M$ for any $M > 0$.

The rest of the hypothesis of Corollary 2.3 can be shown to hold as before. Therefore the solution u can be extended to a solution on all of $\mathbb{R}_+$.

If $\lambda < \lambda_1$ (where again λ_1 is the first eigenvalue of $-\Delta$), we have the asymptotic behavior (stability)

$$\|\Delta u\| \to 0 \quad \text{and} \quad \|u_t\| \to 0 \quad \text{as } t \to \infty.$$

To see this define

$$E_2(t) = \int_\Omega \tfrac{1}{2}((u_t)^2 + (\Delta u)^2 + (\Delta^{-1}[u, u])^2/4 - \lambda u_{x_1}^2 - 2\gamma u u_t - \alpha\gamma u\, \Delta u)\, dx$$

with γ arbitrarily small and to be chosen later. Then,

$$dE_2(t)/dt = -\alpha \|\nabla u_t\|^2 - \gamma \|u_t\|^2 + \langle u, u_{tt}\rangle - \alpha\gamma\langle u, \Delta u\rangle.$$

From the Poincaré inequality, $\|u_t\|^2 \leqslant \lambda_1^{-1} \|\nabla u_t\|^2$ and $\|u_{x_1}\|^2 \leqslant \lambda_1^{-1} \|\Delta u\|^2$ giving us that

$$dE_2(t)/dt \leqslant (-\alpha + \gamma\lambda_1^{-1}) \|\nabla u_t\|^2 - \gamma(1 - \lambda\lambda_1^{-1}) \|\Delta u\|^2.$$

Choosing $\lambda < \lambda_1$ and γ small enough so that $dE_2/dt < 0$ and $E_2 > 0$, we get that $\lim_{t\to\infty} E_2(t)$ exists. Therefore $\|\Delta u\|$ is Cauchy since

$$\int_s^t \|\Delta u(\tau)\|^2 \, d\tau < \int_s^t -(dE_2(\tau)/d\tau)\, d\tau = E_2(s) - E_2(t).$$

Thus $\int_0^\infty \|\Delta u(t)\|^2 \, dt < \infty$ and $\|\Delta u(t)\| \to 0$ as $t \to \infty$. Similarly, $\|u_t\| \to 0$ as $t \to \infty$.

If $\alpha = 0$, we define

$$E_3(t) = \tfrac{1}{2}(\|u_t\|^2 + \|\Delta u\|^2 + \|\Delta^{-1}[u, u]\|^2/4 - \lambda \|u_{x_1}\|^2).$$

$E_3(t) = E_3(0)$ for all t since $dE_3(t)/dt \equiv 0$. Similarly to the above, we can show that $\|\Delta u\|$, $\|u_t\| \leqslant C$ for some positive constant C, independent of t.

Remark. If $\alpha > 0$ and $\lambda \geqslant \lambda_1$, we believe that bifurcation results can be obtained in a manner similar to the damped Klein–Gordon equation of Section 3.

5. The Vibrating Beam Equation

The object of our final example is the study of the nonlinear equation

$$u_{tt} + 2(\alpha\Delta^2 + \delta)\, u_t + \Delta^2 u = f(u) \tag{5.1}$$

in a bounded region $\Omega \subseteq \mathbb{R}^n$ with $\partial\Omega$ smooth. We assume the boundary conditions $u = \Delta u = 0$ on $\partial\Omega$ although other boundary conditions could be used. For $n = 1$ this is the equation of a simply supported vibrating beam where α is viscoelastic structural damping and δ is aerodynamic damping and f is as in cases 1 and 2 below.

We first show that (5.1) is *controlled by an analytic semigroup, contrary to a conjecture formulated by Holmes and Marsden* [5]. We then deal with several nonlinearities one of which involves u_t. Finally, we discuss global continuation, stability, and bifurcation. In particular, we discuss how Galerkin approximations may be used to analyze this equation.

As in Section 3, we pose (5.1) in the complex Hilbert space $L^2(\Omega)$ and let $0<\lambda_1<\cdots$, be the set of eigenvalues for $-\Delta$ and let e_j be the corresponding complete set of orthonormal eigenvectors. Also, as before, $g(\Delta)$ means the closure of $g(\Delta)$ in $L^2(\Omega)$ using the operational calculus associated with the Spectral Theorem [3, p. 1335]. Equation (5.1) is then an ordinary differential equation on $L^2(\Omega)$ with Δ^2 being a positive self-adjoint operator with spectrum $\sigma(\Delta^2)=\{\lambda_j^2;\ -\lambda_j\in\sigma(\Delta)\}$, and $\delta=\delta I$, $I=$ identity. This problem is now equivalent to (2.1) and the operators A_1 and A_2 referred to in Theorem 2.2 are

$$A_k=-\alpha\Delta^2-\delta+(-1)^k(\alpha^2\Delta^4+(2\alpha\delta-1)\,\Delta^2+\delta^2)^{1/2} \tag{5.2}$$

for $k=1,2$. These operators have spectrum

$$\sigma(A_k)=\{(-\nu_j+(-1)^k\,\mu_j),\ j=1,2,...\}$$

for $k=1,2$, and where $\nu_j=(\alpha\lambda_j^2+\delta)$ and $\mu_j=(\alpha^2\lambda_j^4+(2\alpha\delta-1)\,\lambda_j^2+\delta^2)^{1/2}$, $j=1,...$. Hence A_k is a spectral operator in the sense that if $\phi=\sum a_je_j\in D(A_k)$, then $A_k\phi=\sum a_j(-\nu_j+(-1)^k\,\mu_j)\,e_j$, $k=1,2$.

By l'Hospital's rule, it is seen that $\sigma(A_2)$ is bounded and therefore A_2 is a bounded operator. Also $-2(\alpha\lambda_j^2+\delta)\leqslant-\nu_j-\mu_j\leqslant-\alpha\lambda_j^2-\delta$ for j large enough. Hence by the operational calculus for self-adjoint operators, $D(2\alpha\Delta^2+2\delta)\supseteq D(A_1)\supseteq D(\alpha\Delta^2+\delta)$ and thus $D(A_1)=D(\Delta^2)$. Note also that μ_j is real and $-\nu_j-\mu_j$ is negative for all $j\geqslant J$, for some J. Thus A_k generates a semigroup T_k, analytic in the right half plane for $k=1,2$.

We now consider several relevant perturbations.

Case 1. $f(u)=\Gamma\,\Delta u+\bar{\rho}\cdot\nabla u+\kappa\langle u,u\rangle\,\Delta u$, where Γ,κ are constants and $\bar{\rho}$ is a constant vector in $\mathbb{R}^n$. Γ represents in-plane tensile load, $\bar{\rho}$ represents dynamical pressure, and κ represents nonlinear membrane stiffness.

Remark. More generally, f can be any function satisfying

$$\|f(u)\|\leqslant g(K)\qquad\text{and}\qquad\|f(v)-f(u)\|\leqslant g(K)\sum_{j=0}^{2}\|\Delta^j(v-u)\|$$

when $\|\Delta^jv\|$, $\|\Delta^ju\|\leqslant K$, $j=0,1,2$, and g is as in Theorem 2.2. Because of the energy methods used, it is difficult to get global bounds for $\|\Delta^2u\|$, and therefore we tend to consider nonlinearities which satisfy (H1) and (H2) for $B_j=\Delta^j$, $j=0,1$. Nevertheless we use Δ^2 for our local results and observe that for the specific nonlinearities listed above, Δ would be sufficient.

As in the case of the Von Karmán equation, the heart of the matter to prove local existence (and hence global existence if f is linear) is to verify

$$\|\Delta^j(u_{n+1}-u_n)\| \leqslant C\int_0^t \|f_n(s)-f_{n-1}(s)\|\,ds$$

and

$$\|\Delta^j u_{n+1}\| \leqslant C\left(1+\int_0^t \|f_n(s)\|\right)ds$$

for some positive constant C and $j=0, 1, 2$. We prove the second of these for $j=2$ and observe that the other cases follow similarly.

$$\Delta^2 u_{n+1}(t) = \Delta^2 u_0(t) + \phi_n(t),$$

where

$$\phi_n(t) = \Delta^2 \int_0^t \int_s^t T_1(t-\tau)\,T_2(\tau-s)\,f_n(s)\,d\tau\,ds.$$

Rewriting ϕ_n in terms of the eigenvectors e_j we have

$$\phi_n(t) = \sum_{j=1}^{\infty} \lambda_j^2 \int_0^t \int_s^t \exp(-v_j(t-s))\exp((2\tau-t-s)\,\mu_j)\,f_n^j(s)\,d\tau\,ds\,e_j$$

with $f_n^j(s) = \langle f(s, u_n(s)), e_j\rangle$. Thus (for $\mu_j \neq 0$)

$$\begin{aligned}
\|\phi_n(t)\|^2 &= \sum \left\| \frac{\lambda_j^2}{2\mu_j} \int_0^t \exp(-v_j(t-s))[\exp(\mu_j(t-s)) \right. \\
&\qquad \left. -\exp(\mu_j(s-t))\,f_n^j(s)]\,ds\,e_j \right\|^2 \\
&\leqslant \sum \left\| \frac{\lambda_j^2}{2\mu_j} \int_0^t [T_2(t-s)-T_1(t-s)]\,f_n^j(s)\,e_j\,ds \right\|^2 \\
&\leqslant C \left\| \int_0^t (T_2(t-s)-T_1(t-s))\,f_n(s)\,ds \right\|^2 \leqslant \left(C_1 \int_0^t \|f_n(s)\|\,ds\right)^2
\end{aligned}$$

for $C=\sup_j |\lambda_j^2/2\mu_j|$ and C_1 some positive constant. Note that $\mu_j=0$ for at most two values j_1 and j_2. It is easy to handle these terms separately. Thus our claim is established.

Also, from Section 3, we observe that for "nice" initial data, the solution u is a strong solution.

Case 2. $f(u, u_t) = \sigma\langle \Delta u(t), u_t(t)\rangle \Delta u$, where σ is a constant representing nonlinear aerodynamic damping. Here, the local Lipschitz condition on f is

$$\| f(u, u_t)\| \leqslant g(K)$$

and

$$\| f(u, u_t) - f(v, v_t)\| \leqslant g(K)(\|u - v\| + \|\Delta(u - v)\| + \|u_t - v_t\|)$$

when $\|u\|$, $\|v\|$, $\|\Delta u\|$, $\|\Delta v\|$, $\|u_t\|$, $\|v_t\| \leqslant K$ and g is as in (H1). (H2) then needs to be phrased as

$$\|u_n(t)\|, \|\Delta u_n(t)\|, \|u_{n,t}(t)\| \leqslant C + g(K) \int_0^t \| f_n(s)\| \, ds$$

and

$$\begin{aligned}&\|u_{n+1}(t) - u_n(t)\|, \|\Delta(u_{n+1}(t) - u_n(t)\|, \|u_{n+1,t}(t) - u_{n,t}(t)\| \\ &\quad\leqslant g(K) \int_0^t \| f_n(s) - f_{n-1}(s)\| \, ds,\end{aligned}$$

where $t \in [0, T]$, $\|u_n(t)\|$, $\|u_{n-1}(t)\| \leqslant K$, $n = 1, 2,\ldots$, and $f_n(s) = f(s, u_n(s), u_{n,t}(s))$ as in Section 2.

The above bounds can be shown similarly to case 1 except for those on $\|u_{n,t}\|$ and $\|u_{n+1,t} - u_{n,t}\|$. To see these observe (by Lemma 2.1) that

$$\begin{aligned}u_{n+1,t}(t) = u_{0,t}(t) &+ \sigma A_1 \int_0^t \int_s^t T_1(t-\tau)\, T_2(\tau - s)\, f_n(s)\, d\tau\, ds \\ &+ \sigma \int_0^t T_2(t-s)\, f_n(s)\, ds.\end{aligned}$$

Using the procedure in Case 1 and the fact that T_2 is a bounded operator, we get

$$\|u_{n+1,t}(t)\| \leqslant \|u_{0,t}(t)\| + C \int_0^t \| f_n(s)\| \, ds$$

for some positive constant C. Similarly for $\|u_{n+1,t} - u_{n,t}\|$. Thus local solutions exist in this case.

As a corollary of Cases 1 and 2, we therefore have the "vibrating beam" equation with f the sum of the nonlinearities in Cases 1 and 2, has local strong solutions controlled by analytic semigroups.

By a straightforward adaptation of the energies defined in Holmes and Marsden [5] we can show global existence of solutions where f is the sum of the perturbations considered in Cases 1 and 2. In addition, if $\|\bar{\rho}\|^2 < 2(\delta+\alpha\lambda_1^2)(\Gamma+\lambda_1)^{1/2}$ with λ_1 the first eigenvalue of $-\Delta$, we have stability of solutions in the sense that $\|\Delta u\|$, $\|u_t\| \to 0$. To see this consider

$$E(t) = E_1(t) + (\delta + \alpha\lambda_1)\, E_3(t)$$

and (for stability)

$$\bar{E}(t) = E_2(t) + (\delta + \alpha\lambda_1)\, E_3(t),$$

where

$$E_1(t) = \frac{1}{2}\left(\|u_t\|^2 + \|\Delta u\|^2 + \kappa\, \|\nabla u\|^2/2\right),$$

$$E_2(t) = \frac{1}{2}\left(\|u_t\|^2 + \Gamma\, \|\nabla u\|^2 + \|\Delta u\|^2 + \kappa\, \|\nabla u\|^4/2\right),$$

and

$$E_3(t) = \frac{1}{2}\left(\delta\, \|\bar{\rho}\|\, \|u\|^2 + \alpha\, \|\Delta u\|^2 + 2\langle u, u_t\rangle + \frac{\sigma}{2}\, \|\nabla u\|^4\right).$$

With these energies we can prove that $E_t(t) \leqslant CE(t)$ for some $C > 0$ and $\bar{E}_t < 0$, if $\|u\|$ is sufficiently large which, as in Section 4, imply our claims.

Another approach to this problem which is beneficial in studying bifurcation of this and similar types of equations is Galerkin-like approximations. Here we approximate our solution u with a function $u_n = P_n u$, where P_n is the projection onto $\chi_n \equiv \mathrm{span}\{e_1, \ldots, e_n\}$ orthogonal to $\chi_n^\perp \equiv \mathrm{span}\{e_{n+1}, \ldots\}$. Therefore $u_n(t) = \sum_{j=1}^n a_j(t)\, e_j$. The operators A_1, A_2, $\sigma\langle \Delta u(t), u_t(t)\rangle\, \Delta u$, $\Gamma\, \Delta u$, and $\kappa\langle u, u\rangle\, \Delta u$ map $\mathrm{span}\{e_j\}$ onto itself for each $j = 1, \ldots$. Therefore, if $\bar{\rho} = 0$ and the initial data is contained in χ_n, then $u = u_n$, i.e., the vibrating beam equation reduces to a system of second order ordinary differential equations. Using the factoring approach, it can be reduced to a first order system and then the Center Manifold Theorem can be applied as in Section 3.

If $\bar{\rho} \neq 0$, then we replace the operator $\bar{\rho}\cdot\nabla$ with the operator $P_n\bar{\rho}\cdot\nabla$. Then our approximate solution u_n still satisfies an ordinary differential equation and again the Center Manifold Theorem can be applied. In $\mathbb{R}^1$ with $\Omega = [0, 1]$ and $e_j = \sin j\pi x$, $j = 1, \ldots$, these results correspond to those of Holmes and Marsden [5]. The actual calculations in the n-dimensional case are rather messy and lengthy, as expected, but neither illuminating nor surprising. We will content ourselves by saying that under the assumption that $\bar{\rho} = \alpha\bar{\rho}_0$, with α a parameter and $\bar{\rho}_0$ a fixed vector in $\mathbb{R}^n$, the bifurcation results associated to (5.1) are quite similar to those obtained by Holmes and Marsden. Note also that by Corollary 2.4, $u_n(t)$ converges to $u(t)$ uniformly for $t \in [0, T]$ for some $T > 0$. Since the size of T in the proof of Corollary 2.4 was only used to guarantee that u and u_n were uniformly bounded, it is seen that T can be any positive number in this example.

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